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Study of a Prey–Predator Model with Preventing Crop Pest Using Natural Enemies and Control

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Abstract. In this article, we propose a mathematical model of an ecological model that is presented of prey, pest, and natural enemy of pest. Different equilibrium points are obtained, their existence and stability of the steady states are investigated. Further, a mathematical model is obtained in the presence of the control variable for designing optimal pest control problems and to study the effect on crop pest. An optimal control technique is one that minimizes the loss due to crop damage and causes minimum harm to the environment. Several chemicals, biological, physical control, and mechanisms are used. We apply Pontryagin's maximum principle to find the existence, characterization, and necessary conditions of the optimal control. Numerical simulations are further verified to support analytical results and to illustrate a different system that can be observed in the model.

INTRODUCTION

In the last 50 years, the world population has risen at an unprecedented pace. New agricultural technology has increased the production of major crops yet, besides causing deterioration of the climate. However, there is a lack of effective data on food loss or damage caused by such biotic agents, mainly in developing nations. The small available information demonstrates that an estimated 18-20% of yearly food crops globally may be damaged by pests. Moreover, losses are comparatively higher in the emerging tropics of Asia and Africa, at which population is also projected to grow rapidly over the next 50 years. There is a significant need to accurately measure the level of waste production at various levels from farming to consumer use. It is vital to the advancement of secure, productive, and sustainable methods of pest control and food safety for the long term. More recently, the topic of the management of agricultural insects has received more and more research interest. Unscientific and unjustified application of insecticides has led to catastrophic results. It has been noticed that 10-30% of yield reductions are attributed to pests. The changing agro–climatic factors have contributed significantly to the infestation of several predatory insects in different areas of the world. Approximately 30–35% of the annual agricultural productivity loss in India is attributed to pests, as reported by the Indian Agricultural Research Council. In recent days, nematodes have become a serious risk to crops and lead to the loss of 60 million tons of grains annually worldwide. This has contributed to a detrimental effect on agricultural biosecurity, which is critical to food security. The output of agricultural crops is affected by the occurrence of pests. Numerous methods exist to avoid or reduce these crop losses. The various types of crop waste as well as the various pest control techniques evolved over the last century have been summarized below. An outline of grain declines is provided for wheat, maize, rice, soybeans, potatoes, and cotton for the period 2001-03 on both a local and worldwide basis, despite the current crop protection methods. The cumulative pest loss ranged from around 50% in wheat to 80% in cotton yield. It is expected that yield reduction for wheat, cotton, and soybean is about 26–29%, and 31%, 37 %, and 40%, respectively, for rice, maize, and potatoes. Lastly, weeds generated the highest possible liability (34%). Cultivation security in cash crops was much more beneficial than in food crops. While the use of pesticides has helped farmers to increase crop yield, it has also increased susceptibility to the pest's harmful effects. The definition of integrated pest/crops aims to find the optimal standard for the use of pest control methods and to decrease the quantity or frequency of pesticides to an environmentally safe degree. At the global level [17], about one-third of the total grain production has been lost due to insects, viruses, and weeds. Arthropods are killing about 18-20% percent of annual crops, worth more than US$470 billion worldwide. Big problems (13–16%) exist in the areas due to disease, and the declines in underdeveloped nations are higher. Globally around 27 to 32% of all agricultural food produce in the world are wasted [18]. But these losses did not include the agricultural crop losses in the field. For example Table 1 shows the loss of crops attributable to insect plagues in India. The data shown in Table 1 taken from first row references.

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TABLE 1. Reports of crop damage caused by pests (%) in India

Crop	[2]	[20]	[7]	[6]	[29]	[19]	[11]	[15]
Oilseeds	5	5	35	25	25	25	15	20
Cotton	18	18	50	22	50	50	30	30
Pulses	5	5	30	7	15	15	15	15
Rice	10	10	25	18.6	25	25	25	25
Wheat	3	-	5-10	11.4	5	5	5	5
Maize	5	-	25	-	25	25	20	18
Sorghum and millets	3.5	-	35	10	30	30	10	8
Groundnut	5	-	15	-	15	15	15	15
Sugarcane	10	-	20	15	20	20	20	20

Agricultural crop damages caused by pests in India were also reported from periodically [2, 10] and, after the green revolution, the increase in crop losses was higher than that reported at the global stages [2, 10]. The crop wastage rose significantly from 7.2% percent in the 1960s to 23.3% in the early 2000s, but after that, the losses fell to 17.5% [11]. In the pre-green revolution period, it was reported that losses incurred by pest species ranged from 3.5 % in sorghum and millets to 16% in cotton. Vegetable crop damage was estimated to be very massive in India, which was around 30–40 % [16]. In addition to harvest and post-harvest losses, which were reported to be 10–30 % of production, pre–harvest yield reduction of about 40% was unavoidable [14]. As calculated by [11], yield losses declined from 23.3% in the 1990s to 17.5% in 2010 and 15.7% recently [15]. [12] reported a real loss of 39% caused by pests in potatoes globally and without crop security about 71% can be lost to pests. Actual overall losses have been reported to range from 24% in Europe to over 50% in Africa. The yield losses in Indian vegetables due to huge insects are given in Table 2 [13, 16].

TABLE 2. Loss of yields in Indian vegetable crops caused by large pest species

Crop	Pest	Yield loss (%)
Chilli Thrips	Chilli Thrips	12–90
	Mites (Polyphagotarsonemus latus)	34
Tomato	Fruit borer (H. armigera)	24–73
Okra	Fruit borer (H. armigera)	22
	Leafhopper (A. biguttula biguttula)	54–66
	Whitefly (B. tabaci)	54
	Shoot and fruit borer (E. vittella)	23–54
Cabbage	Diamondback moth (P. xylostella)	17–99
	Cabbage caterpillar (P. brassicae)	69
	Cabbage leaf webber	28–51
	Cabbage borer (H. undalis)	30–58
Cabbage	Aphid	3–6
	Tobacco caterpillar	4–8
	Potato tuber moth	6–9
	Mite (P. latus)	4–27
Cucurbits	Fruit fly (B. cucurbitae)	20–100
Brinjal	Fruit and shoot borer	11–93

Because of the animals, pests, diseases, and weeds at least one–third to half of the world crop output is destroyed. Reducing the loss of food would lead to a major increase in the supply of food for use. A precise estimation of these damages is the first important step in reducing these losses. However, it is acknowledged that damage is rising due to different biotic and abiotic stresses in the face of the growing strength of agriculture and the environment on farmland. Thus, there is an urgent need to establish effective pest control techniques and agricultural pests enemy that is at the same time productive, healthy, and persistent. Using pest–resistant cultivars provides so many benefits and can form the center in which to build sustainable farming production. Pest control has appeared recently being one of the most important problems impacting ecologically sustainable due to the growing people communities and increasing food crisis. Hence, there is detailed research of effective pest control strategies. Prey–predation mechanism of prey disease and stage-structured model behaviour are studied by [36, 37, 38]. [35] built up a prey-predator model of the Lokta–volta model kind with irrational impact for integrated pest control. [39] researched a model that consists of predator species, susceptible prey, and infected prey and two pushes for integrated control of the pest. The optimal management technique was explored by [40] using a balanced mix of controls (infection, chemical, and predation pest control aspects) to avoid the pest population. The [42, 43] suggested a state-dependent dynamic impulsive mechanism by releasing natural enemies and sprinkling pesticides. [44] has mentioned a pest–natural enemy model focused on the application of impulsive pest-specific pesticides. Relevant studies were performed using various kinds of prey–predator models and stage–structured populations. The goal of the present research is to increase crop production and to reduce the pest population. For this, at the higher trophic level of the pest species, we find an appropriate predator(the natural enemy of pest). The study seeks to determine how natural enemy use and balance pesticide can manage the pests. Also, the aim is to illustrate how the monitoring of pest control is greatly influenced by the deletion of the natural enemy due to predation by its predators or the use of dangerous chemical substances.

THE MODEL

Many species like spiders, birds, frogs, etc. are recognized as the natural enemy as they feed on agricultural pests. Biological monitoring is the beneficial of predators operation for controlling pests and their serious harm. Through the number of insects and mites, the natural enemies could be used as a biological control for pest management. The use of natural enemies is also useful for the biological management of the land area and wildland weeds. High levels of residues from pesticides can interfere with the reproduction of natural enemies and they are trying to identify and destroy pests. Therefore, the elimination of pests and the protection of the natural enemy from an agriculture sector and ecological viewpoint is very important. By using pesticides, apply them selectively and handle only highly infested areas instead of whole plants. Use more common insecticides in the kinds of invertebrates they destroy, like Bacillus thuringiensis, which only destroys caterpillars that consume treated foliage. Here, we find the prey–predator model [41] since it is renowned for displaying excitable activity and expanding this model by introducing another equation of the prey-treated crop species. In this model, The mathematical model is set up to protect crops from pests through the effective use of natural enemies along with the balanced application of pesticides. To examine the effect of the natural enemy on control of the pest species, we first find a framework of prey–predator in three dimensions. The following is a purely ecological model:

$$\frac{dP_1}{dt} = r_1P_1(1-\frac{P_1}{K_1}) - \alpha_1P_1P_2, \tag{1}$$

$$\frac{dP_2}{dt} = r_2P_2(1-\frac{P_2}{K_2}) - \frac{mP_3P_2^2}{\alpha_2^2+P_2^2} + \alpha_3P_1P_2, \tag{2}$$

$$\frac{dP_3}{dt} = \frac{\gamma P_3P_2^2}{\alpha_2^2+P_2^2} - \mu P_3, \tag{3}$$

with initial data $P_1(0) > 0$, $P_2(0) > 0$ and $P_3(0) > 0$. Context of parameters is shown in the Table 3.

TABLE 3. Parameters used in the model

Parameter	Description
$P_1(t)$	Density of crops (prey) at time t
$P_2(t)$	Density of pest at time t
$P_3(t)$	Density of natural enemies of pest at time t
r_1	Intrinsic growth rate of prey
K_1	The carrying capacity of prey
r_2	Intrinsic growth rate of pest
K_2	The carrying capacity of pest
m	Holling type-III functional response with consumption rate
α_1	Predation rate coefficient
α_2	Half saturation constant
α_3	Reproduction rate of pest per prey eaten
γ	Predator conversion rate
μ	The natural mortality rate of the predator

The predation loss term $\frac{mP_3P_2^2}{\alpha_2^2+P_2^2}$ has a moving property in the pest equation. We also assumed that the natural enemy is removed from the ecosystem due to predation by its predator apart from natural death. In population ecology, the natural enemies of pest population has different functional responses. We assume that the overall functional response is Holling type III. The carrying capacity K_1 is the number of prey population that can fill a unit area in the absence of pest,$K2$is the population density of pest capable of filling a unit area in the nonexistence of predators and m indicates that the natural enemy may eat the available pest per unit time by converting γ of that quantity into a predator. Also, μ means that through natural death, the predator population dies per unit time. Finally, δ is the rate of natural enemy removal.

BOUNDEDNESS OF THE SYSTEM

Theorem 1. *The solutions of the system ((1), (2), (3)) in the region* $\mathbb{R}^2_+$ *are bounded.*

Proof. We establish the function

$$w(P_1,\, P_2,\, P_3) = P_1 + P_2 + P_3.$$

The derivative of the $w(P_1,\, P_2,\, P_3)$ with respect to time t is

$$\begin{aligned}
\frac{dw}{dt} =& \frac{dP_1}{dt} + \frac{dP_2}{dt} + \frac{dP_3}{dt} \\
=& r_1P_1(1-\frac{P_1}{K_1}) - \alpha_1P_1P_2 + r_2P_2(1-\frac{P_2}{K_2}) - \frac{mP_3P_2^2}{\alpha_2^2+P_2^2} + \alpha_3P_1P_2 + \frac{\gamma P_3P_2^2}{\alpha_2^2+P_2^2} - \mu P_3 \\
\frac{dw}{dt} \leq& r_1P_1(1-\frac{P_1}{K_1}) + r_2P_2(1-\frac{P_2}{K_2}) - \mu P_3 \\
\frac{dw}{dt} + \tau w \leq& r_1P_1(1-\frac{P_1}{K_1}) + r_2P_2(1-\frac{P_2}{K_2}) - \mu P_3 + \tau(P_1+P_2+P_3) \\
\frac{dw}{dt} + \tau w \leq& r_1P_1(1-\frac{P_1}{K_1}) + r_2P_2(1-\frac{P_2}{K_2}) - \mu P_3 + \tau(P_1+P_2+P_3) \\
=& (r_1+\tau)P_1 + (r_2+\tau)P_2 - \frac{r_1}{K_1}P_1^2 - \frac{r_2}{K_2}P_2^2 - (\mu-\tau)P_3 \\
\frac{dw}{dt} + \tau w \leq& \frac{K_1(r_1+\tau)^2}{4r_1} + \frac{K_2(r_2+\tau)^2}{4r_2} = L,
\end{aligned}$$

here $0 < \tau < \mu$ and $L = \frac{K_1(r_1+\tau)^2}{4r_1} + \frac{K_2(r_2+\tau)^2}{4r_2}$. Using application of Gronwalls inequality [22], we obtain

$$0 < w(t) \leq e^{-\tau t}(w(0) - \frac{L}{\tau}) + \frac{L}{\tau},$$

when $t \to \infty$, yields $0 < w(t) < \frac{L}{\tau}$. □

LOCAL STABILITY ANALYSIS

Equilibria of the system

The model (1), (2) and (3) has the following equilibrium points
(i) The trivial equilibrium $E_0(P_1 = 0,\, P_2 = 0,\, P_3 = 0)$, which always exists.
(ii) $E_1(P_1 = 0,\, P_2 = K_2,\, P_3 = 0)$.
(iii) $E_2\,(P_1 = K_1,\, P_2 = 0,\, P_3 = 0)$.
(iv) $E_3\left(P_1 = 0,\, P_2 = -\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}},\, P_3 = \frac{\gamma r_2\alpha_2^2\left(\sqrt{\gamma-\mu}K_2+\sqrt{\mu}\alpha_2\right)}{m(\gamma-\mu)^{3/2}K_2}\right)$.

(v) $E_4\left(P_1=0,\, P_2=\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}},\, P_3=\frac{\gamma r_2\alpha_2^2\left(\sqrt{\gamma-\mu}K_2-\sqrt{\mu}\alpha_2\right)}{m(\gamma-\mu)^{3/2}K_2}\right)$.
(vi) $E_5\left(P_1=\frac{K_1r_2(r_1-K_2\alpha_1)}{r_1r_2+K_1K_2\alpha_1\alpha_3},\, P_2=\frac{K_2r_1(r_2+K_1\alpha_3)}{r_1r_2+K_1K_2\alpha_1\alpha_3},\, P_3=0\right)$.
(vii) $E_6\left(P_1=K_1\left(1+\frac{\sqrt{\mu}\alpha_1\alpha_2}{\sqrt{\gamma-\mu}r_1}\right), P_2=-\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}},\, P_3=\frac{\gamma\alpha_2^2\left(\sqrt{\mu}r_1r_2\alpha_2+K_2\left(\sqrt{\mu}K_1\alpha_1\alpha_2\alpha_3+\sqrt{\gamma-\mu}r_1(r_2+K_1\alpha_3)\right)\right)}{m(\gamma-\mu)^{3/2}K_2r_1}\right)$.
(viii) $E_7\left(P_1=K_1\left(1-\frac{\sqrt{\mu}\alpha_1\alpha_2}{\sqrt{\gamma-\mu}r_1}\right), P_2=\frac{\sqrt{\mu}\alpha_2}{\sqrt{\gamma-\mu}},\, P_3=\frac{\gamma\alpha_2^2\left(-\sqrt{\mu}r_1r_2\alpha_2+K_2\left(-\sqrt{\mu}K_1\alpha_1\alpha_2\alpha_3+\sqrt{\gamma-\mu}r_1(r_2+K_1\alpha_3)\right)\right)}{m(\gamma-\mu)^{3/2}K_2r_1}\right)$.

Local Stability

The equilibrium stability of smooth differential equations is evaluated by the sign of a real part of eigenvalues of the Jacobian matrix. The Jacobian matrix is the matrix of the partial derivatives of the right–hand side with respect to state variables where all derivatives are assessed at the equilibrium points and its eigenvalues decide stability. An equilibrium is asymptotically stable if all eigenvalues have negative real parts; it is unstable if there is at least one positive real part in eigenvalues of the Jacobian matrix. Moreover, the Routh–Hurwitz stability criterion uses the characteristic polynomial of a Jacobian matrix to provide stability information. Now, to examine the local stability of equilibrium points, the Jacobian matrix $J(P_1,\, P_2,\, P_3)$ of the prey–pest–natural enemy of system ((1), (2), (3)) at any equilibrium point $(P_1,\, P_2,\, P_3)$ is evaluated as

$$J(P_1,\, P_2,\, P_3)=\begin{pmatrix} -\frac{P_1r_1}{K_1}+\left(1-\frac{P_1}{K_1}\right)r_1-P_2\alpha_1 & -P_1\alpha_1 & 0 \\ P_2\alpha_3 & -\frac{P_2r_2}{K_2}+\left(1-\frac{P_2}{K_2}\right)r_2+\frac{2mP_2^2P_3}{(P_2^2+\alpha_2^2)^2}-\frac{mP_3}{P_2^2+\alpha_2^2}+P_1\alpha_3 & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & -\frac{2\gamma P_2^3P_3}{(P_2^2+\alpha_2^2)^2}+\frac{2\gamma P_2P_3}{P_2^2+\alpha_2^2} & -\mu+\frac{\gamma P_2^2}{P_2^2+\alpha_2^2} \end{pmatrix}.$$

(i)The Jacobian matrix at equilibrium point $E_0(P_1=0,\, P_2=0,\, P_3=0)$ is

$$\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -\mu \end{pmatrix}$$

and the eigen values at the E_0 is r_1, r_2 and $-\mu$. Therefore the equilibrium point is saddle.
(ii) The Jacobian matrix for the equilibrium point $E_1(P_1=0,\, P_2=K_2,\, P_3=0)$ is

$$\begin{pmatrix} r_1-K_2\alpha_1 & 0 & 0 \\ K_2\alpha_3 & -r_2 & -\frac{mK_2}{K_2^2+\alpha_2^2} \\ 0 & 0 & -\mu+\frac{\gamma K_2^2}{K_2^2+\alpha_2^2} \end{pmatrix}.$$

The eigenvalues of $J(E_1)$ are $-r_2,\, r_1-K_2\alpha_1$ and $\frac{\gamma K_2^2-\mu K_2^2-\mu\alpha_2^2}{K_2^2+\alpha_2^2}$. Therefore the equilibrium point $E_1(P_1=0,\, P_2=K_2,\, P_3=0)$, is locally asymptotically stable if $r_1-K_2\alpha_1<0$ and $\frac{\gamma K_2^2-\mu K_2^2-\mu\alpha_2^2}{K_2^2+\alpha_2^2}<0$.
(iii) The Jacobian matrix for the equilibrium point $E_2(P_1=K_1,\, P_2=0,\, P_3=0)$ is

$$\begin{pmatrix} -r_1 & -K_1\alpha_1 & 0 \\ 0 & r_2+K_1\alpha_3 & 0 \\ 0 & 0 & -\mu. \end{pmatrix}.$$

The eigenvalues of $J(E_2)$ are $-\mu,\, -r_1$ and $r_2+K_1\alpha_3$. Since two eigenvalues are negative and other is always positive, therefore the system around the (E_2) point is saddle point.
(iii) The Jacobian matrix for the equilibrium point $E(P_1=0,\, P_2\neq 0,\, P_3\neq 0)$ is

$$\begin{pmatrix} r_1-P_2\alpha_1 & 0 & 0 \\ P_2\alpha_3 & -\frac{P_2r_2}{K_2}+\frac{2mP_2^2P_3}{(P_2^2+\alpha_2^2)^2}-\frac{mP_3}{P_2^2+\alpha_2^2}+\frac{mP_3P_2}{P_2^2+\alpha_2^2} & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & -\frac{2\gamma P_2^3P_3}{(P_2^2+\alpha_2^2)^2}+\frac{2\gamma P_2P_3}{P_2^2+\alpha_2^2} & 0 \end{pmatrix}.$$

The characteristic polynomial of jacobian matrix is

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

where, $a_1 = \left(-r_1 + \frac{P_2 r_2}{K_2} + P_2\alpha_1 - \frac{2mP_2^2 P_3}{(P_2^2+\alpha_2^2)^2} + \frac{mP_3}{P_2^2+\alpha_2^2} - \frac{mP_2 P_3}{P_2^2+\alpha_2^2}\right)$,

$a_2 = \left(-\frac{P_2 r_1 r_2}{K_2} + \frac{P_2^2 r_2 \alpha_1}{K_2} + \frac{2mP_2^2 P_3 r_1}{(P_2^2+\alpha_2^2)^2} - \frac{2mP_2^3 P_3 \alpha_1}{(P_2^2+\alpha_2^2)^2} - \frac{mP_3 r_1}{P_2^2+\alpha_2^2} + \frac{mP_2 P_3 r_1}{P_2^2+\alpha_2^2} + \frac{mP_2 P_3 \alpha_1}{P_2^2+\alpha_2^2} - \frac{mP_2^2 P_3 \alpha_1}{P_2^2+\alpha_2^2}\right)$, $a_3 = 0$.

For the local asymptotically stable of the system, the following Routh-Hurwitz criterion must be satisfied: (1) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, (2) $a_1 a_2 - a_3 > 0$.

(iv) The Jacobian matrix for the equilibrium point $E(P_1 \neq 0,\ P_2 = 0,\ P_3 \neq 0)$ is

$$J = \begin{pmatrix} -\frac{P_1 r_1}{K_1} & -P_1\alpha_1 & 0 \\ 0 & r_2 - \frac{mP_3}{\alpha_2^2} + P_1\alpha_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $J(E)$ are 0, $-\frac{P_1 r_1}{K_1}$ and $r_2 - \frac{mP_3}{\alpha_2^2} + P_1\alpha_3$. In this case, system is not local asymptotically stable.

(v) The Jacobian matrix for the equilibrium point $E(P_1 \neq 0,\ P_2 \neq 0,\ P_3 = 0)$ is

$$\begin{pmatrix} -\frac{P_1 r_1}{K_1} & -P_1\alpha_1 & 0 \\ P_2\alpha_3 & -\frac{P_2 r_2}{K_2} + \frac{mP_3 P_2}{P_2^2+\alpha_2^2} & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & 0 & -\mu + \frac{\gamma P_2^2}{P_2^2+\alpha_2^2} \end{pmatrix}.$$

The characteristic polynomial of jacobian matrix is

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

where $a_1 = \left(\mu + \frac{P_1 r_1}{K_1} + \frac{P_2 r_2}{K_2} - \frac{\gamma P_2^2}{P_2^2+\alpha_2^2} - \frac{mP_2 P_3}{P_2^2+\alpha_2^2}\right)$

$a_2 = \left(\frac{\mu P_1 r_1}{K_1} + \frac{\mu P_2 r_2}{K_2} + \frac{P_1 P_2 r_1 r_2}{K_1 K_2} + \frac{m\gamma P_2^3 P_3}{(P_2^2+\alpha_2^2)^2} - \frac{m\mu P_2 P_3}{P_2^2+\alpha_2^2} - \frac{\gamma P_1 P_2^2 r_1}{K_1\left(P_2^2+\alpha_2^2\right)} - \frac{mP_1 P_2 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)} - \frac{\gamma P_2^3 r_2}{K_2\left(P_2^2+\alpha_2^2\right)} + P_1 P_2 \alpha_1 \alpha_3\right)$

$a_3 = \frac{\mu P_1 P_2 r_1 r_2}{K_1 K_2} + \frac{m\gamma P_1 P_2^3 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)^2} - \frac{m\mu P_1 P_2 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)} - \frac{\gamma P_1 P_2^3 r_1 r_2}{K_1 K_2\left(P_2^2+\alpha_2^2\right)} + \mu P_1 P_2 \alpha_1 \alpha_3 - \frac{\gamma P_1 P_2^3 \alpha_1 \alpha_3}{P_2^2+\alpha_2^2}$.

For the local asymptotically stable of the system, the following Routh–Hurwitz criterion must be satisfied:
(1) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$,
(2)$a_1 a_2 - a_3 > 0$.

(vi) The Jacobian matrix for the equilibrium point $E(P_1 \neq 0,\ P_2 \neq 0,\ P_3 \neq 0)$ is

$$\begin{pmatrix} -\frac{P_1 r_1}{K_1} & -P_1\alpha_1 & 0 \\ P_2\alpha_3 & -\frac{P_2 r_2}{K_2} + \left(1 - \frac{P_2}{K_2}\right) r_2 + \frac{2mP_2^2 P_3}{(P_2^2+\alpha_2^2)^2} - \frac{mP_3}{P_2^2+\alpha_2^2} + P_1\alpha_3 & -\frac{mP_2}{P_2^2+\alpha_2^2} \\ 0 & -\frac{2\gamma P_2^3 P_3}{(P_2^2+\alpha_2^2)^2} + \frac{2\gamma P_2 P_3}{P_2^2+\alpha_2^2} & 0 \end{pmatrix}.$$

The characteristic equation of system around the equilibrium is

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

where $a_1 = \left(\frac{P_1 r_1}{K_1} + \left(-1 + \frac{2P_2}{K_2}\right) r_2 - \frac{2mP_2^2 P_3}{(P_2^2+\alpha_2^2)^2} + \frac{mP_3}{P_2^2+\alpha_2^2} - P_1\alpha_3\right)$

$a_2 = \left(P_1\left(-1 + \frac{2P_2}{K_2}\right) r_1 \frac{r_2}{K_1} - \frac{2m\gamma P_2^4 P_3}{(P_2^2+\alpha_2^2)^3} + \frac{2m\gamma P_2^2 P_3}{(P_2^2+\alpha_2^2)^2} - \frac{2mP_1 P_2^2 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)^2} + \frac{mP_1 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)} - \frac{P_1^2 r_1 \alpha_3}{K_1} + P_1 P_2 \alpha_1 \alpha_3\right)$

$a_3 = \frac{2m\gamma P_1 P_2^4 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)^3} + \frac{2m\gamma P_1 P_2^2 P_3 r_1}{K_1\left(P_2^2+\alpha_2^2\right)^2}$.

For the local asymptotically stable of the system, the following Routh–Hurwitz criterion must be satisfied:
(i) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$,
(ii)$a_1 a_2 - a_3 > 0$.

THE MODEL IN THE PRESENCE OF CONTROL

Lately, pest-control has become a remarkable issue in ecological field as it can reduce the losses of both food and economy. Recent studies and research has shown that pest population can be decreased by using natural enemy of pest and chemicals like pesticides. It is noticed that spraying of pesticides had reduced the growth of pest population initially but the pest population may revival rapidly after a period of little time or larger than pre treatments level in later stages. Recently, [40] considered the mixture of controls based on prey infection, predation and chemical pesticide application and implemented an optimal plan for monitoring. It is importance that our model can also be modified by using balance pesticides as a control variable when pesticides kill pests. Thus, our new model is

$$\frac{dP_1}{dt} = r_1P_1(1-\frac{P_1}{K_1}) - \alpha_1P_1P_2, \tag{4}$$

$$\frac{dP_2}{dt} = r_2P_2(1-\frac{P_2}{K_2}) - \frac{mP_3P_2^2}{\alpha_2^2+P_2^2} + \alpha_3P_1P_2 - ubP_2, \tag{5}$$

$$\frac{dP_3}{dt} = \frac{\gamma P_3P_2^2}{\alpha_2^2+P_2^2} - \mu P_3. \tag{6}$$

with initial data $P_1(0) > 0,\ P_2(0) > 0$ and $P_3(0) > 0$. Here, natural enemies of pest P_3 have been used as a biological control strategy for removing harmful organism species. Biological organism control helps to find the natural enemies of a pest and implementing such natural predators into the specific community where the pest has accumulated. The natural enemies must be before release to establish that they do not have a detrimental effect on the climate or the economy. Furthermore, many entomopathogens are widely viable in a dosage form that can be treated like a pesticide. Term $\frac{mP_3P_2^2}{\alpha_2^2+P_2^2}$ is negative impact on harmful pests in agriculture. We presume that functional response of type III happens between pest species and their natural enemies, as survival first increases with increasing density of pests, and then decreases. When pest density is controlled by biological control and pesticides, crop density increases. Therefore, P_1 and P_3 are not taken in minimize the objective functional. In addition, pesticide regulation u should concentrate on time and should be used as needed. While our main goal throughout this section is to minimize the number of pests by using pesticide. The aim is to minimize the pest density of P_1. To attain the objective, through this way we construct the functional objective of our optimal control problem as regards:

$$J(u) = \min_u \int_0^{T_f} (A_1P_2(t) + A_2u(t)^2)dt, \tag{7}$$

where $A_1 > 0$ and $A_2 > 0$ are weights. A_1 is corresponding to the pest density and A_2 is the parameter of the square of control. The harmful side outcome of the pesticide are remove by the square of the control [21, 32, 33]. The objective functional is subject to ((4), (5) (6)). The term A_2u^2 is the cost of control on rate of application of pesticide strategy. We want to find a optimal control such that

$$J(u^*) = \min\{J(u)|u \in U\}, \tag{8}$$

where U is control set defined by below:

$$U = \{u(t)|u(t) \text{ are lebesgue measurable with } 0 \le t \le 1,\ t \in [0,T].\} \tag{9}$$

Based on its relative significance, the weight of state variables is typically allocated while many of the control are allocated according to their expense impacts. With the help of Pontryagins maximum principle [25], we obtain the requisite criteria to evaluate the value of the control such that the J is optimized. If this feasible control occurs then this is recognized as optimal control [26, 27]. By establishing the Hamiltonian H and then introducing the maximum Pontryagin principle, We establish the necessary criteria for the existence of optimal control. Adjoint functions also have a common theme in multivariate calculus as Lagrange multipliers, which add constraints to the function of multiple variables to be optimized. Thus, we consider finding suitable requirements that the adjoint function should fulfill. Then, by separating the diagram from control to functional objective. For minimization, the control variables are strongly convex in the Hamiltonian equation. Let the Hamiltonian H is

$$H(t,P(t),\lambda(t),u(t)) = f(t,P(t),u(t)) + \lambda(t)g(t,P(t),u(t)),$$

where $f(t,P(t),u(t))$ is integrand of the functional $J(u)$ and $g(t,P(t),u(t))$ is right hand side of the equation (4), (5) and (6). If $P_1^*(t), P_2^*(t), P_3^*(t)$ and $u^*(t)$ is an optimal solution, then there is a non–trivial vector function $\lambda(t)$ satisfying the following equations:

$$\begin{cases} \frac{dP}{dt} = \frac{\partial H(t,P^*(t),\lambda(t),u^*(t))}{\partial \lambda}, \\ 0 = \frac{\partial H(t,P^*(t),\lambda(t),u^*(t))}{\partial u}, \\ \lambda'(t) = -\frac{\partial H(t,P^*(t),\lambda(t),u^*(t))}{\partial P}. \end{cases}$$

If the control is bounded, i.e. $a \leq u(t) \leq b$, then optimal control $u^*(t)$ [27] is given by

$$\begin{cases} u^* = a, & \text{if } \frac{\partial H}{\partial u} < 0, \\ a \leq u^* \leq b, & \text{if } \frac{\partial H}{\partial u} = 0, \\ u^* = b, & \text{if } \frac{\partial H}{\partial u} > 0. \end{cases}$$

Thus, to find the optimal value of J, we make the Hamiltonian as:
$H = A_1P_2 + A_2u^2 + \lambda_1(r_1P_1(1-\frac{P_1}{K_1}) - \alpha_1P_1P_2) + \lambda_2(r_2P_2(1-\frac{P_2}{K_2}) - \frac{mP_3P_2^2}{\alpha_2^2+P_2^2} + \alpha_3P_1P_2 - ubP_2) + \lambda_3(\frac{\gamma P_3P_2^2}{\alpha_2^2+P_2^2} - \mu P_3)$
with $\lambda = [\lambda_1,\ \lambda_2,\ \lambda_3]^T$, the adjoint vector associated to the state P_1, P_2 and P_3 associated to the prey-predator model. Pontryagin's maximum principle calculate the exact evolution of the adjoint functions. The optimality scheme of equations can be computed by taking partial derivatives of the Hamiltonian corresponding to the state variable. The optimality technique consists of the state and adjoint systems along with the characterization of the control variables.

Theorem .2. *Given the objective functional* $J(u) = \min_u \int_0^{T_f} (A_1P_2(t) + A_2u(t)^2)dt$, *with initial data* $P_1(0) = P_{10} > 0$, $P_2(0) = P_{20} > 0$, $P_3(0) = P_{30} > 0$, $t \in [0, T_f]$ *and* $u(t)$ *is measurable with* $0 \leq u(t) \leq 1$, *then there is an optimal control* $u^*(t)$ *which minimizes the objective functional* $J(u^*)$ *such that*

$$J(u^*) = \min\{J(u) | 0 \leq u(t) \leq 1\}$$

.

Proof. To investigate the existence of optimal control, we must first show the following result using the boundedness of the system over a finite given period. The following condition should be fulfilled:
(i) The set of state variables and controls is non–empty.
(ii) The control set U is closed and convex.
(iii) Objective functional (integrand) is convex in U.
(iv) The state system R.H.S is bounded by a linear function in the variables of state and control where coefficients depend on the state and time.
(v) The integrand bounded below by $c_1(|u|^2)^{\frac{\beta}{2}} - c_2$ for $c_1 > 0$, $c_2 > 0$ and $\beta > 1$. We use the results from [27, 28] to check the above conditions. The state and control variables are non–negative values, and in the minimization problem, the convexity of the necessary condition of the objective functional in $u(t)$ is met. The control set is convex and closed by definition and the admissible Lebesgue measurable control variables set $u(t)$ is also convex and closed, integrand $f(t,p,u) = A_1P_2 + A_2u^2$ is clearly convex in u.

For the forth condition, The system is linear in the control and can be expressed as

$$\overrightarrow{g}(t, \overrightarrow{P}, u) = \overrightarrow{\alpha}(t, \overrightarrow{P}) + \overrightarrow{\beta}(t, \overrightarrow{P})u,$$

where $\overrightarrow{g}(t, \overrightarrow{P}, u)$ is R.H.S of the state equation, $\overrightarrow{P} = (P_1, P_2, P_3)$, $\overrightarrow{\alpha}$ and $\overrightarrow{\beta}$ is a vector valued function of $\overrightarrow{P}$. Using solution of bounded, we see that

$$\left|\left[\ \overrightarrow{g}(t,\overrightarrow{P},u)\ \right]\right| \le \left|\left[\begin{array}{ccc} r_1 & 0 & 0 \\ \alpha_3 P_2 & r_2 & 0 \\ 0 & 0 & \gamma \end{array}\right]\right|\left|\left[\begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array}\right]\right| + \left|\left[\begin{array}{c} 0 \\ -b_2 u P_2 \\ 0 \end{array}\right]\right|$$

$$\le \left|\left[\begin{array}{ccc} r_1 & 0 & 0 \\ \alpha_3 K_2 & r_2 & 0 \\ 0 & 0 & \gamma \end{array}\right]\right|\left|\left[\begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array}\right]\right| + \left|\left[\begin{array}{c} 0 \\ b_2 u K_2 \\ 0 \end{array}\right]\right|$$

$$\le C_1|\overrightarrow{P}| + C_2|u|,$$

where C_1 is dependent on system coefficients and C_2 is maximum norm value.
For the last condition (5),

$$A_1P_2 + A_2u^2 \ge A_2(|u|^2)^{\frac{\beta}{2}} \ge c_1(|u|^2)^{\frac{\beta}{2}} - c_2,$$

where c_1 and $c_2 > 0$, Hence, there exists constants c_1, c_2 and $\beta > 1$ such that $A_1P_2 + A_2u^2 \ge c_1(|u|^2)^{\frac{\beta}{2}} - c_2$.
The optimal system satisfies all the above conditions and it has been bounded closed, which determines the compactness for the existence of the optimal control. □

Theorem .3. *If $u^*(t)$ be an optimal control which minimize $J(u)$. Let $P_1^*(t)$, $P_2^*(t)$ and $P_3^*(t)$ are optimal state solutions for the control system* (4)*,* (5) *and* (6)*, then there exist adjoint variables $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ such that:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -(\frac{\partial H}{\partial P_1}) = -\frac{(K_1-2P_1)r_1\lambda_1}{K_1} + P_2(\alpha_1\lambda_1 - \alpha_3\lambda_2), \\
\frac{d\lambda_2}{dt} &= -(\frac{\partial H}{\partial P_2}) = -A_1 + bu\lambda_2 - r_2\lambda_2 + \frac{2P_2r_2\lambda_2}{K_2} - \frac{2mP_2^2P_3\lambda_2}{(P_2^2+\alpha_2^2)^2} + \frac{mP_3\lambda_2}{P_2^2+\alpha_2^2} + P_1(\alpha_1\lambda_1 - \alpha_3\lambda_2) + \frac{2\gamma P_2^3P_3\lambda_3}{(P_2^2+\alpha_2^2)^2} - \frac{2\gamma P_2P_3\lambda_3}{P_2^2+\alpha_2^2}, \\
\frac{d\lambda_3}{dt} &= -(\frac{\partial H}{\partial P_3}) = \frac{mP_2\lambda_2 + (-\gamma+\mu)P_2^2\lambda_3 + \mu\alpha_2^2\lambda_3}{P_2^2+\alpha_2^2},
\end{aligned} \tag{10}$$

with the transversality conditions are

$$\lambda_i(T_f) = 0\, for\ \ i = 1,\ 2,\ 3. \tag{11}$$

Further, the optimal control variable u^* that minimize $J(u)$ is given by

$$u^*(t) = min\,\{max\,\{0,\ \frac{\lambda_2 bP_2^*}{2A_2}\},\ 1\}. \tag{12}$$

Proof. The transversality and adjoint conditions are given result by [27, 45], using the Hamiltonian function H:

$$H(t,\ \overrightarrow{P},\ \overrightarrow{u}) = f(t,\ \overrightarrow{P},\ \overrightarrow{u}) + \sum_{i=1}^{3}\lambda_i g_i(t,\ \overrightarrow{P},\ \overrightarrow{u}),$$

where $f(t,\ \overrightarrow{P},\ \overrightarrow{u})$ is integrand of the functional and $g_i(t,\ \overrightarrow{P},\ \overrightarrow{u})$ are state equations.

$$H = A_1P_2 + A_2u^2 + \lambda_1(r_1P_1(1-\tfrac{P_1}{K_1}) - \alpha_1P_1P_2) + \lambda_2(r_2P_2(1-\tfrac{P_2}{K_2}) - \tfrac{mP_3P_2^2}{\alpha_2^2+P_2^2} + \alpha_3P_1P_2 - ubP_2) + \lambda_3(\tfrac{\gamma P_3P_2^2}{\alpha_2^2+P_2^2} - \mu P_3)$$

with $\lambda = [\lambda_1,\ \lambda_2,\ \lambda_3]^T$, the adjoint vector associated to the state P_1, P_2 and P_3. The Pontryagin's maximum principle [1] provides adjoint system which can be followed as:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -(\frac{\partial H}{\partial P_1}) = -\frac{(K_1-2P_1)r_1\lambda_1}{K_1} + P_2(\alpha_1\lambda_1 - \alpha_3\lambda_2), \\
\frac{d\lambda_2}{dt} &= -(\frac{\partial H}{\partial P_2}) = -A_1 + bu\lambda_2 - r_2\lambda_2 + \frac{2P_2r_2\lambda_2}{K_2} - \frac{2mP_2^2P_3\lambda_2}{(P_2^2+\alpha_2^2)^2} + \frac{mP_3\lambda_2}{P_2^2+\alpha_2^2} + P_1(\alpha_1\lambda_1 - \alpha_3\lambda_2) + \frac{2\gamma P_2^3P_3\lambda_3}{(P_2^2+\alpha_2^2)^2} - \frac{2\gamma P_2P_3\lambda_3}{P_2^2+\alpha_2^2}, \\
\frac{d\lambda_3}{dt} &= -(\frac{\partial H}{\partial P_3}) = \frac{mP_2\lambda_2 + (-\gamma+\mu)P_2^2\lambda_3 + \mu\alpha_2^2\lambda_3}{P_2^2+\alpha_2^2}.
\end{aligned}$$

The transversality conditions are

$$\lambda_i(T_f) = 0 \text{ for } i = 1, 2, 3.$$

The optimality condition provides

$$\begin{aligned}\frac{\partial H}{\partial u} &= 2A_2u - \lambda_2 bP_2 = 0 \; at \; u = u^*(t)\\ &\Rightarrow 2A_2u^* - \lambda_2 bP_2 = 0 = 0\\ &\Rightarrow u^*(t) = \frac{.\lambda_2 bP_2^*}{2A_2}.\end{aligned}$$

□

NUMERICAL SIMULATION

For the simulation objective, we take a simulated set of parameters for minimize pest population and maximize the crop population and we take the initial population for the prey, pest and natural enemies as 2, 1.2, and 0.8 respectively. Fig. 1 explore the dynamics of the system (1), (2) and (3) by using numerical simulations and computations. Next, we obtain the solution of optimal control problem numerically using fourth order Runge-Kutta procedure [27, 31, 32, 33, 34]. Fig. 2 shows the solution curves of the state variables crop(prey), pest and natural enemy population in the presence of control and natural enemies versus in the absence of control and natural enemies. Observation shows that the application of optimal control variables and natural enemy reduce a quite larger number of pest population compared to in the absence of the control and natural enemy. Crop population increases from 2 $gram\ cm^{-3}$ to more than 5.5 $gram\ cm^{-3}$ approximately at $t = 2\ months$. Again from the Fig. 2, it is easy to observe that pest population increases rapidly in the absence of control and natural enemies and almost constant in the presence of control and natural enemies. Fig. 2(c) and Fig. 2(d) represent the variation of natural enemy and control variables with respect to time respectively. In Fig. 4, there are some changes in parameters like weight $A_1 = 1$ to 100, $A_2 = 1$ to 50, $r_1 = 0.9\ month^{-1}$ to $0.5\ month^{-1}$, $r_2 = 0.258\ month^{-1}$ to $0.35\ month^{-1}$, $\alpha_2 = 0.25; gram\ cm^{-3}$ to $0.2; gram\ cm^{-3}$, $b = 1$ to 2, $t = 2\ months$ to $0.5\ months$. Fig. 4(a) shows the crop production increase from 2 $gram\ cm^{-3}$ to 2.25 $gram\ cm^{-3}$ at $t = 0.5$ months, Fig. 4(b) represent the pest population variation in the presence and absence of natural enemy and pesticide.

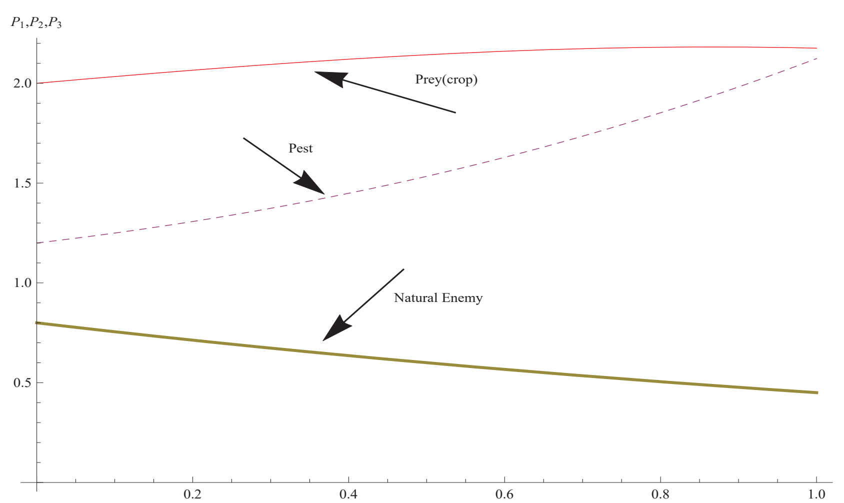

FIGURE 1. This figure illustrates the dynamics of the system between prey, pest and natural enemy of pest with respect to the time, where values of parameters are $r_1 = 0.5\ month^{-1}, K_1 = 20\ gram\ cm^{-3}, m = 0.7\ month^{-1}$, $\alpha_1 = 0.23\ gram^{-1}\ cm^3\ month^{-1}, r_2 = .35\ month^{-1}, K_2 = 15\ gram\ cm^{-3}, \alpha_2 = 0.3\ gram\ cm^{-3}, \alpha_3 = .25\ gram^{-1}\ cm^3\ month^{-1}$, $\gamma = .026\ month^{-1}$, $\mu = 0.6\ month^{-1}$, $P_{10} = 2\ gram\ cm^{-3}$, $P_{20} = 1.2\ gram\ cm^{-3}$, $P_{30} = 0.9\ gram\ cm^{-3}$ and $t = 2\ months$.

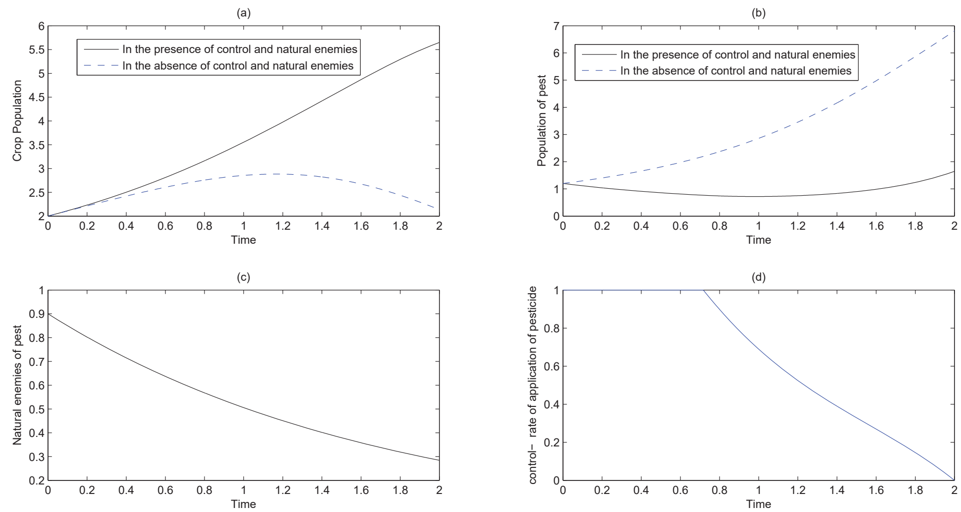

FIGURE 2. This graph depicts the results of the comparison between the curves of prey(crops), pest and predator individuals in the presence of control and natural enemies versus in the absence of control and natural enemies, natural enemies population and control variable with respect to the time, where values of parameters are $A_1 = 1 \ month^{-1}$, $A_2 = 1$, $r_1 = 0.9$, $K_1 = 20 \, gram \, cm^{-3}$, $\alpha_1 = 0.23 \ gram^{-1} \, cm^3 \, month^{-1}$, $r_2 = 0.258 \, month^{-1}$, $K_2 = 15 \, gram \, cm^{-3}$, $m = 0.7 \, month^{-1}$, $\alpha_2 = 0.25 \, gram \, cm^{-3}$, $\alpha_3 = 0.258 \ gram^{-1} \, cm^3 \, month^{-1}$, $b = 1$, $\gamma = 0.0256 \, month^{-1}$, $\mu = 0.6 \, month^{-1}$, $P_{10} = 2 \, gram \, cm^{-3}$, $P_{20} = 1.2 \, gram \, cm^{-3}$, $P_{30} = 0.9 \, gram \, cm^{-3}$ and $t = 2 \, months$.

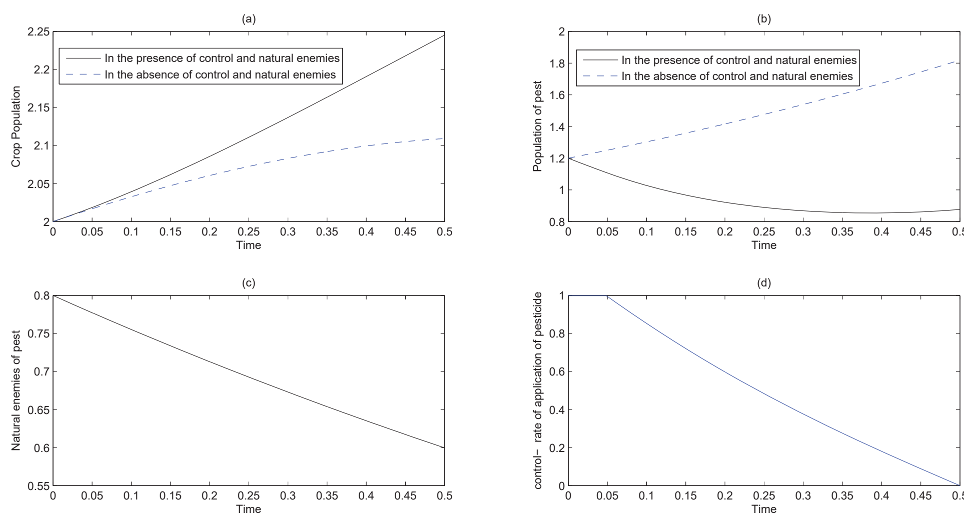

FIGURE 3. This figure depicts the results of the comparison between the curves of prey(crops), pest and predator population in the presence of control and natural enemies versus in the absence of control and natural enemies, natural enemies population and control variable with respect to the time, where values of parameters are $A_1 = 100$, $A_2 = 50$, $r_1 = 0.5 \, month^{-1}$, $K_1 = 20 \, gram \, cm^{-3}$, $\alpha_1 = 0.23 \ gram^{-1} \, cm^3 \, month^{-1}$, $r_2 = 0.35 \, month^{-1}$, $K_2 = 15 \, gram \, cm^{-3}$, $m = 0.7 \, month^{-1}$, $\alpha_2 = 0.2 \, gram \, cm^{-3}$, $\alpha_3 = 0.25 \ gram^{-1} \, cm^3 \, month^{-1}$, $b = 2$, $\gamma = 0.026 \, month^{-1}$, $\mu = 0.6 \, month^{-1}$, $P_{10} = 2 \, gram \, cm^{-3}$, $P_{20} = 1.2 \, gram \, cm^{-3}$, $P_{30} = 0.8 \, gram \, cm^{-3}$ and $t = 0.5 \, months$.

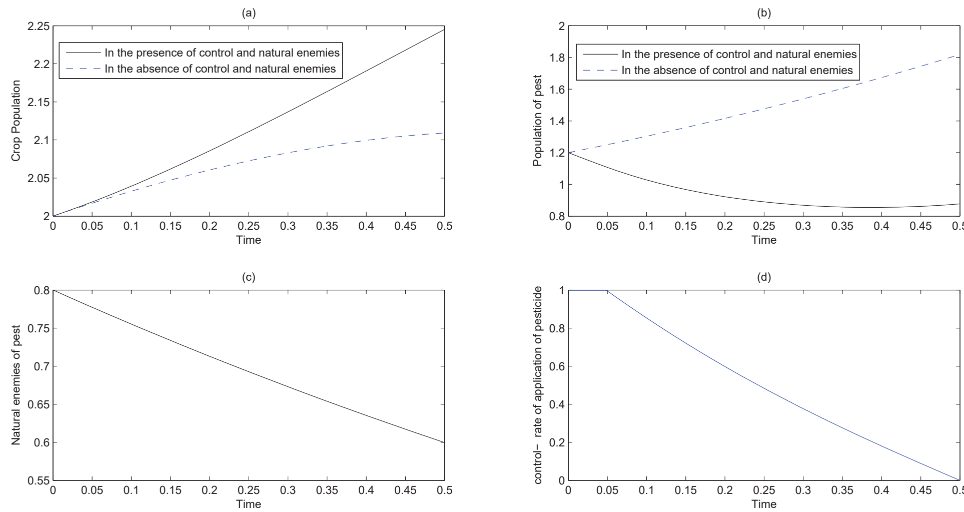

FIGURE 4. This figure depicts the results of the comparison between the curves of prey(crops), pest and predator.

CONCLUSION

In this study, we proposed a three-dimensional model of prey(crop), pest(a predator of prey), and natural enemy of pest(a predator of pest) and a systematic approach has been applied to control the pest population using a combination of pesticide and a natural enemy of pest. The studies have shown the system is bound. The existence and stability criteria of the equilibrium point have been established. Further, by applying the Pontryagins maximum principle, we proved the existence and determined the necessary conditions of the optimal control. We have also numerically analyzed the significantly affect of the natural enemy on the pest population and crop production in the model. In the presence of the natural enemy and control variables, the pest population has been decreased and crop(prey) production has been increased in the prey–pest–natural enemy of pest model. Thus, our results showed that the natural enemy has a significant role in regulating the pest population.

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