Bull. Aust. Math. Soc. **111** (2025), 107–117
doi:10.1017/S0004972724000431

ENUMERATION OF GROUPS IN SOME SPECIAL VARIETIES OF A-GROUPS

ARUSHI and GEETHA VENKATARAMAN

(Received 3 March 2024; accepted 29 April 2024; first published online 27 August 2024)

Abstract

We find an upper bound for the number of groups of order n up to isomorphism in the variety $\mathfrak{S} = \mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$, where p, q and r are distinct primes. We also find a bound on the orders and on the number of conjugacy classes of subgroups that are maximal amongst the subgroups of the general linear group that are also in the variety $\mathfrak{A}_q\mathfrak{A}_r$.

2020 *Mathematics subject classification*: primary 20E10; secondary 20B15, 20B35, 20D10, 20E45, 20H30.

Keywords and phrases: group enumeration, variety of groups, general linear group, symmetric group, transitive subgroup, primitive subgroup, conjugacy class.

1. Introduction

A group is an A-group if its nilpotent subgroups are abelian. For any class of groups $\mathfrak{B}$, we denote the number of groups of order n up to isomorphism by $f_{\mathfrak{B}}(n)$. Computing $f(n)$ becomes harder as n gets bigger. Thus, in the area of group enumerations, we attempt to approximate $f(n)$. When counting is restricted to the class of abelian groups, A-groups or groups in general, the asymptotic behaviour of $f(n)$ varies significantly. Let $f_{A,sol}(n)$ be the number of isomorphism classes of soluble A-groups of order n. Dickenson [2] showed that $f_{A,sol}(n) \leq n^{c \log n}$ for some constant c. McIver and Neumann [7] showed that the number of nonisomorphic A-groups of order n is at most $n^{\lambda+1}$, where λ is the number of prime divisors of n including multiplicities. In the same paper, they stated the following conjecture based on a result of Higman [4] and Sims [12] on p-group enumerations.

CONJECTURE 1.1. Let $f(n)$ be the number of (isomorphism classes of groups of) order n. Then $f(n) \leq n^{(2/27+\epsilon)\lambda^2}$, where $\epsilon \to 0$ as $\lambda \to \infty$.

In 1993, Pyber [9] proved a powerful version of Conjecture 1.1: the number of groups of order n with specified Sylow subgroups is at most $n^{75\mu+16}$, where μ is the